

Vectors

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Vectors are important in game development, especially in the areas of physics and 3D graphics. You can use vectors to represent a game object's position in the game world, its velocity, and its acceleration. Vectors are also used to respond to collisions between objects. 3D graphics uses vectors for numerous operations, including lighting calculations, bump mapping, and environment mapping. This article introduces you to vectors and vector math.

Introduction

Before defining what a vector is, I need to define what a scalar quantity is. A scalar quantity has magnitude only. Examples of scalar quantities are 51, 1.34, and -12.56 . A vector quantity has both magnitude and direction. Traveling 35 miles per hour on a motorcycle is a scalar quantity. Traveling 35 miles per hour north is a vector quantity.

Vector quantities use the sign to signify direction. The vector quantities 5 and -5 are the same vector with opposite directions, which you can see in Figure 1.

The most common vectors used in computer programs are 2D and 3D vectors. 2D vectors have two components: x and y. 3D vectors have three components: x, y, and z. I'm going to use 3D vectors for the examples in this article. If you can understand 3D vectors, you'll have no trouble with 2D vectors.

Vector Length and Unit Vectors

To calculate a vector's length, square the vector's components, add the squares, and take the square root of the sum, which you can see in the following equation:

$$\begin{aligned} \mathbf{v} &= \langle x, y, z \rangle \\ \|\mathbf{v}\| &= \sqrt{(x * x) + (y * y) + (z * z)} \end{aligned}$$

You may also see a vector's length referred to as the norm or the magnitude. The following example calculates the length of a 3D vector:

$$\begin{aligned} \mathbf{v} &= \langle 9, 2, 6 \rangle \\ \|\mathbf{v}\| &= \sqrt{(9 * 9) + (2 * 2) + (6 * 6)} = \sqrt{81 + 4 + 36} = 11 \end{aligned}$$

A unit vector is a vector with a length of 1. Normalizing a vector converts a vector to a unit vector. To normalize a vector, divide each component by its length. Let's normalize the vector from the previous example.

$$\mathbf{u} = \langle 9/11, 2/11, 6/11 \rangle = \langle .818, .182, .545 \rangle$$

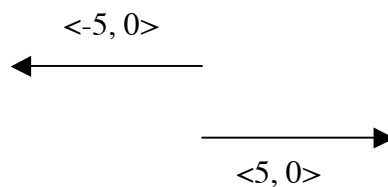


Figure 1

Two vectors with the same magnitude and opposite directions.

Arithmetic Operations on Vectors

You can perform many of the same arithmetic operations on vectors that you can on scalar quantities. Adding two vectors involves adding the second vector's components to the first vector's components.

$$\begin{aligned} \mathbf{v}_1 &= \langle x_1, y_1, z_1 \rangle & \mathbf{v}_2 &= \langle x_2, y_2, z_2 \rangle \\ \mathbf{v}_1 + \mathbf{v}_2 &= \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle \end{aligned}$$

If you have two vectors, \mathbf{v}_1 and \mathbf{v}_2

$$\mathbf{v}_1 = \langle 7, -2, 4 \rangle \qquad \mathbf{v}_2 = \langle 1, 5, 6 \rangle$$

The sum of \mathbf{v}_1 and \mathbf{v}_2 is

$$\mathbf{v}_1 + \mathbf{v}_2 = \langle 7 + 1, -2 + 5, 4 + 6 \rangle = \langle 8, 3, 10 \rangle$$

Subtracting two vectors involves subtracting each of the second vector's components from the first vector's components.

$$\begin{aligned} \mathbf{v}_1 &= \langle x_1, y_1, z_1 \rangle & \mathbf{v}_2 &= \langle x_2, y_2, z_2 \rangle \\ \mathbf{v}_1 - \mathbf{v}_2 &= \langle x_1 - x_2, y_1 - y_2, z_1 - z_2 \rangle \end{aligned}$$

If you have two vectors, \mathbf{v}_1 and \mathbf{v}_2

$$\mathbf{v}_1 = \langle 3, 7, -1 \rangle \qquad \mathbf{v}_2 = \langle 4, 2, 5 \rangle$$

Subtracting \mathbf{v}_2 from \mathbf{v}_1 yields

$$\mathbf{v}_1 - \mathbf{v}_2 = \langle 3 - 4, 7 - 2, -1 - 5 \rangle = \langle -1, 5, -6 \rangle$$

Next section I cover the dot product, which is the result of multiplying two vectors. You can easily multiply a vector by a scalar value. If you multiply a vector \mathbf{v}

$$\mathbf{v} = \langle x, y, z \rangle$$

by a scalar s , you get the result

$$\mathbf{v} * s = \langle s * x, s * y, s * z \rangle$$

Multiplying the vector \mathbf{v}

$$\mathbf{v} = \langle -2, 5, 7 \rangle$$

by the scalar value 5 gives you the result

$$\langle -2 * 5, 5 * 5, 7 * 5 \rangle = \langle -10, 25, 35 \rangle$$

Dot Product

The dot product multiplies two vectors and produces a scalar quantity. Multiply each of the first vector's components by the second vector's corresponding component. Add the products of each component to get the dot product.

$$\begin{aligned} \mathbf{v}_1 &= \langle x_1, y_1, z_1 \rangle & \mathbf{v}_2 &= \langle x_2, y_2, z_2 \rangle \\ \mathbf{v}_1 \cdot \mathbf{v}_2 &= (x_1 * x_2) + (y_1 * y_2) + (z_1 * z_2) \end{aligned}$$

Let's look at an example.

$$\begin{aligned} \mathbf{v}_1 &= \langle -5, 3, 6 \rangle & \mathbf{v}_2 &= \langle 4, 5, 7 \rangle \\ \mathbf{v}_1 \cdot \mathbf{v}_2 &= (-5 * 4) + (3 * 5) + (6 * 7) = -20 + 15 + 42 = 37 \end{aligned}$$

An alternative method of calculating the dot product is to multiply the lengths of the two vectors and multiply the product by the cosine of the interior angle (See Figure 2) between the two vectors.

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = ||\mathbf{v}_1|| * ||\mathbf{v}_2|| * \cos(\theta)$$

You wouldn't want to calculate the dot product using the second formula because it's slower and more complicated to calculate. The second formula helps to calculate the interior angle.

$$\cos(\theta) = (\mathbf{v}_1 \cdot \mathbf{v}_2) / (||\mathbf{v}_1|| * ||\mathbf{v}_2||)$$

Cross Product

The cross product of two 3D vectors creates a vector that is perpendicular to the plane where the two vectors lie. A normal is a vector that is perpendicular to a surface, such as a plane. The cross product lets you calculate normal vectors. Use the following formula to calculate the cross product:

$$\begin{aligned} \mathbf{v}_1 &= \langle x_1, y_1, z_1 \rangle & \mathbf{v}_2 &= \langle x_2, y_2, z_2 \rangle \\ \mathbf{v}_1 \times \mathbf{v}_2 &= \langle x_3, y_3, z_3 \rangle \\ x_3 &= (y_1 * z_2) - (z_1 * y_2) \\ y_3 &= (z_1 * x_2) - (x_1 * z_2) \\ z_3 &= (x_1 * y_2) - (y_1 * x_2) \end{aligned}$$

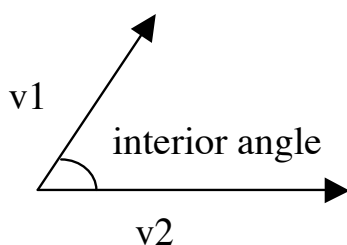


Figure 2

The interior angle between two vectors.

As you can see from its formula, the cross product is more complicated to calculate than the other operations covered in this article. Let's look at an example.

$$\begin{aligned}v_1 &= \langle -5, 3, 6 \rangle & v_2 &= \langle 4, 5, 7 \rangle \\x_3 &= (3 * 7) - (6 * 5) = 21 - 30 = -9 \\y_3 &= (6 * 4) - (-5 * 7) = 24 - (-35) = 59 \\z_3 &= (-5 * 5) - (3 * 4) = -25 - 12 = -37 \\v_1 \times v_2 &= \langle -9, 59, -37 \rangle\end{aligned}$$

Parallel and Orthogonal Vectors

Two vectors are parallel if one of the vectors is a scalar multiple of the other. The following vectors are parallel:

$$v_1 = \langle 2, 3, 4 \rangle \quad v_2 = \langle 6, 9, 12 \rangle$$

If you multiply vector v_1 by 3, you get vector v_2 , which makes the two vectors parallel.

Orthogonal means perpendicular. Two vectors are orthogonal if their dot product is 0. Use the cross product to calculate an orthogonal 3D vector. Calculating an orthogonal vector for a 2D vector is easier. If you have a 2D vector v

$$v = \langle x, y \rangle$$

There are two vectors orthogonal to v .

$$u_1 = \langle y, -x \rangle \quad u_2 = \langle -y, x \rangle$$

Two vectors are orthonormal if they're both unit vectors (have length 1) and are orthogonal.

Conclusion

As I finish I want you to understand the terms norm, normal, normalize, and orthonormal. Norm is the vector's length. A normal is a vector that is perpendicular to a surface. Normalizing a vector converts it to a unit vector, a vector of length 1. Two vectors are orthonormal if they are unit vectors and are perpendicular to each other.

There is source code to accompany this article. The source code consists of C++ classes for 2D and 3D vectors. If you have any questions or comments about this article, send them to mark-AT-meandmark-DOT-com.